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SNAPPING OF AN ELASTIC PSEUDO-RIGID BODY

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Abstract—In this paper, we are concerned with the finite deformation of a neo-Hookean hyperelastic pseudo-rigid membrane with a hinged support point and with a movable point on a fixed straight line. The movable point is subjected to a dead load, and the distance between the support point and the line is arbitrary. After determining the static equilibrium positions of the membrane, the elastic properties of the structure are indicated. Finally, snapping behavior of the membrane is described by means of the theory of elastic stability. The results obtained may be applied to representing a secant modulus of shear for an unstable elastic material point.

1. INTRODUCTION

Static postbuckling and snapping of thin structures have been the subject of intensive research from the 1950s, because of their wide applications in civil, mechanical and aerospace engineering. A numerical work on circular arches can be found in the article of Huddleston (1968). A detailed review on the relevant problem of arches, beams and rings was given in the surveys of DaDeppo and Schmidt (1970) and Schmidt and DaDeppo (1971). As for shells, the research papers are also too numerous to be listed. Here, we cite only one due to Brodland and Cohen (1987). Because it is difficult to solve the non-linear differential equations that characterize the behavior of these thin structures, numerical methods are usually adopted. On the other hand, the static stability of homogeneously-deformed elastic cubes and elastic square sheets, subject to symmetric loads, has been investigated in analytic ways by, for example, Rivlin (1948), Beatty (1965, 1967) and MacSithigh (1986). However, up to now, we have not found research work dealing with the snapping of a homogeneously-deformed elastic body with dimension more than one. For this reason, the static snapping of a neo-Hookean hyperelastic pseudo-rigid membrane (a two-dimensional body) will be studied in this paper, based on the theory of elastic stability. The same approach of analysis may be generalized to a hyperelastic pseudo-rigid disc (a three-dimensional body).

A body is pseudo-rigid if its deformation gradient is homogeneous throughout the body whatever the loads. This is analogous to a rigid body whose deformation gradient is orthogonal whatever the loads. For the theory of pseudo-rigid bodies, cf. the monograph of Cohen and Muncaster (1988). As a matter of fact, when a body is regarded to be pseudo-rigid, one encounters only ordinary differential equations or algebraic equations, respectively, when dynamic motion or static deformation of the body is to be determined. Of course, it is clear that in a static problem a homogeneous deformation may be exact, while in a dynamic problem a homogeneous deformation is merely approximate. With the application of the theory of pseudo-rigid bodies, the solutions of a variety of dynamic problems of elastic bodies have been obtained by Cohen and Muncaster (1988), Cohen and MacSithigh (1989) and Cohen and Sun (1988, 1990). We now explain another possible application of the theory of pseudo-rigid bodies. Since deformations of a pseudo-rigid body are homogeneous, it behaves mechanically and mathematically like a material point. Considering that there is no restriction to the shape of a pseudo-rigid body, we may use it to model a material point with arbitrary configuration. It is known that some material points exhibit complex properties, such as elastic instability. By analyzing elastically-unstable deformations of a pseudo-rigid body, one may gain insight into the unstable behavior of a material point, and use this to establish a constitutive relation for an elastically-unstable material point. This is another reason for us to investigate the snapping of an elastic

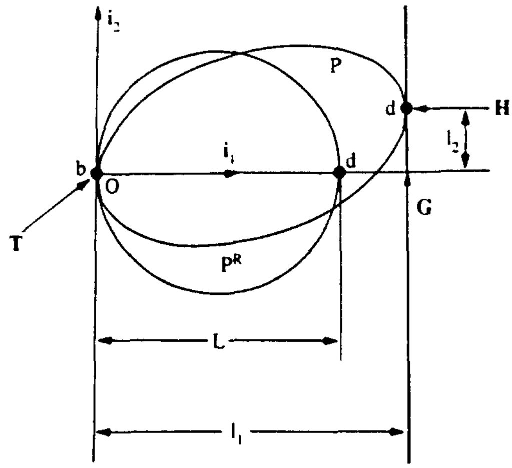

Fig. 1. A pseudo-rigid membrane with a fixed point b and a movable point d on a fixed straight line.

membrane. Regarding the membrane as a pseudo-rigid body, we are able to find an analytic solution denoting the relation between load and deformation that includes snapping—a phenomenon of elastic instability.

In Section 2, we first emphasize that for a pseudo-rigid body, its center of mass can be regarded as a material point, and can be chosen as a base point of the body. Then we review the balance equations of linear momentum and tensor moment of momentum for a pseudo-rigid body. In Section 3, the problem of a flat pseudo-rigid membrane with a hinged support point and a movable point on a fixed straight line is presented, as shown in Fig. 1. A dead load is applied at the movable point of the membrane. The distance between the fixed point and the line is arbitrary. The deformation of the membrane is allowed to be finite, while the material of the membrane is assumed to be neo-Hookean hyperelastic. The equations that govern the static deformation of the structure are given at the end of Section 3. In Section 4, a perturbation solution for a small displacement of the membrane subject to a small dead load is derived. For finite deformation of the membrane, the properties of the structure are qualitatively presented in terms of the relation between the load and the displacement of the movable point. Then numerical examples are given to explain the behavior of the structure. In Section 5, based on the energy criteria (the adjacent method and the derivative method) of elastic stability, stable and unstable equilibrium positions of the membrane are indicated to illustrate snapping behavior of the membrane. With the results obtained, we may use an elastic membrane to construct a "spring" with positive, negative and neutral elasticity, as well as the snapping behavior indicated in Sections 4 and 5. At the end of Section 5, we explain that our results may also be used to represent a secant modulus of shear for an unstable elastic material point.

2. PRELIMINARIES OF PSEUDO-RIGID BODIES

We take a three-dimensional Euclidean space E with a fixed origin o to model an inertial frame of reference. Let P^R and P denote the reference and the present configurations of a continuous material body in E. We suppose the body to undergo a smooth motion χ such that the positions $\mathbf{X} \in P^R$ and $\mathbf{x} \in P$ of a material point p in the body are related by

$$\mathbf{x}(p) = \chi(\mathbf{X}(p), t), \tag{1}$$

where t is the time parameter. When ρ^R, ρ and m are used to denote the mass densities measured on P^R, P and the total mass of the body, respectively, conservation of mass can be represented by the global form:

$$m = \int_{P^R} \rho^R = \int_P \rho. \tag{2}$$

The centers of mass of P^R and P, respectively, are defined by

$$\mathbf{X}_c := \frac{1}{m}\int_{P^R} \rho^R \mathbf{X}, \qquad \mathbf{x}_c := \frac{1}{m}\int_P \rho \mathbf{x}. \tag{3}$$

A body B is said to be *pseudo-rigid* if in all motions its deformation gradient

$$\mathbf{F} := \partial \mathbf{x}/\partial \mathbf{X} \qquad \text{with} \qquad \det \mathbf{F} > 0 \tag{4}$$

depends only on time t. For the more detailed definition, the reader may refer to the monograph of Cohen and Muncaster (1988). It is obvious that a general motion satisfying (4) can be represented as

$$\mathbf{x} = \mathbf{x}_0(t) + \mathbf{F}(t)\mathbf{X}, \tag{5}$$

where $\mathbf{x}_0 \in E$ is an arbitrary vector. Now let $\mathbf{X}_b$ and $\mathbf{x}_b$ be the referential and the present positions of a material point b in B, respectively. Then it follows from (5) that for a pseudo-rigid body, the following two relations hold:

$$\mathbf{x}_b = \mathbf{x}_0(t) + \mathbf{F}(t)\mathbf{X}_b, \tag{6}$$

$$\mathbf{x} = \mathbf{x}_b(t) + \mathbf{F}(t)(\mathbf{X} - \mathbf{X}_b). \tag{7}$$

Since the point b serves to describe the motion, it will be called a *base point* of the body B in what follows.

It is evident that if b is a fixed point of the body B, then

$$\mathbf{x}_b(t) = \mathbf{X}_b \tag{8}$$

for all time. In this case, a motion of the body B can be written as

$$\mathbf{x} = (\mathbf{I} - \mathbf{F}(t))\mathbf{X}_b + \mathbf{F}(t)\mathbf{X}, \tag{9}$$

from (7) and (8), where $\mathbf{I}$ is the identity tensor. In particular, if the origin o of E is chosen in such a way that it coincides with the fixed point b, then $\mathbf{X}_b = \mathbf{0}$, which together with (9) yields

$$\mathbf{x} = \mathbf{F}(t)\mathbf{X}. \tag{10}$$

Furthermore, by integration of (7) over B with the density of mass as a weighting function and by the use of (2) and (3), we obtain

$$\mathbf{x}_c = \mathbf{x}_b(t) + \mathbf{F}(t)(\mathbf{X}_c - \mathbf{X}_b). \tag{11}$$

A comparison of (7) and (11) shows that for a pseudo-rigid body, its center of mass undergoes the same motion with a material point in it, although the center of mass of the body may not be a real material point. This property of a pseudo-rigid body allows us to treat the center of mass as a material point. As an application of this property, we are able to choose the center of mass as a base point of the body. This application then can be extended to a system of multi-pseudo-rigid bodies or a rod with homogeneously-deformed cross-sections. If a pseudo-rigid body has a fixed point, then we have

$$\mathbf{x}_c = (\mathbf{I}-\mathbf{F}(t))\mathbf{X}_b+\mathbf{F}(t)\mathbf{X}_c. \tag{12}$$

In addition, if the choice for (10) is taken, then

$$\mathbf{x}_c = \mathbf{F}(t)\mathbf{X}_c. \tag{13}$$

Here, validity of both (12) and (13) is based on the fact that the center of mass can be regarded as a material point for a pseudo-rigid body.

In order to determine a motion of a pseudo-rigid body B, we need the following balance equations of linear momentum and tensor moment of momentum described on P^R and P, respectively, given by Cohen and Muncaster (1988):

$$\mathbf{f}^R = m\ddot{\mathbf{x}}_c, \qquad \mathbf{M}^R-\mathbf{\Sigma}^R = \ddot{\mathbf{F}}\mathbf{E}^R; \tag{14}$$

$$\mathbf{f} = m\ddot{\mathbf{x}}_c, \qquad \mathbf{M}-\mathbf{\Sigma} = (\dot{\mathbf{L}}+\mathbf{L}^2)\mathbf{E}. \tag{15}$$

Here,

$$\mathbf{f}^R = \int_{S^R}\mathbf{t}^R+\int_{P^R}\rho^R\mathbf{b}^R, \qquad \mathbf{f} = \int_S \mathbf{t}+\int_P \rho\mathbf{b}; \tag{16}$$

$$\mathbf{M}^R = \int_{S^R}\mathbf{t}^R\otimes\mathbf{X}^*+\int_{P^R}\rho^R\mathbf{b}^R\otimes\mathbf{X}^*,$$

$$\mathbf{M} = \int_S \mathbf{t}\otimes\mathbf{x}^*+\int_P\rho\mathbf{b}\otimes\mathbf{x}^*; \tag{17}$$

$$\mathbf{\Sigma}^R = \int_{P^R}\mathbf{S}^R, \qquad \mathbf{\Sigma} = \int_P \mathbf{S};$$

$$\mathbf{E}^R = \int_{P^R}\mathbf{X}^*\otimes\mathbf{X}^*\rho^R, \qquad \mathbf{E} = \int_P \mathbf{x}^*\otimes\mathbf{x}^*\rho; \tag{18}$$

$$\mathbf{X}^* = \mathbf{X}-\mathbf{X}_b, \qquad \mathbf{x}^* = \mathbf{x}-\mathbf{x}_b, \qquad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{19}$$

with b as either the center of mass, or a fixed point of B if it exists; $\mathbf{S}^R$ and $\mathbf{S}$ are the Piola–Kirchhoff stress and the Cauchy stress so that $\mathbf{\Sigma}^R$ and $\mathbf{\Sigma}$ are the internal force-moments in P^R and P, respectively, and both of them are constitutive quantities; $\mathbf{t}^R$ and $\mathbf{t}$ are tractions on $S^R = \partial(P^R)$ and $S = \partial(P)$; and $\mathbf{b}^R$ and $\mathbf{b}$ are body forces per unit mass of P^R and P, respectively. It follows from the above definitions that $\mathbf{f}^R$, $\mathbf{M}^R$ and $\mathbf{f}$, $\mathbf{M}$ are the external forces, the external force-moments with respect to the point b, acting on P^R and P, respectively; and $\mathbf{E}^R$ and $\mathbf{E}$ are the Euler tensors of P^R and P with respect to $\mathbf{X}_b$ and $\mathbf{x}_b$, respectively. In fact, (14) and (15) can be transformed from each other through the following relations:

$$\mathbf{f} = \mathbf{f}^R, \qquad \mathbf{M} = \mathbf{M}^R\mathbf{F}^T, \qquad \mathbf{E} = \mathbf{F}\mathbf{E}^R\mathbf{F}^T, \qquad \mathbf{\Sigma} = \mathbf{\Sigma}^R\mathbf{F}^T. \tag{20}$$

Cohen and Sun (1988) have pointed out that if an arbitrary material point of the body is chosen as a base point, the balance equations of tensor moment of momentum will not have the simple forms $(14)_2$ and $(15)_2$.

In the theory of pseudo-rigid bodies described above, the mechanical property of the body is not specified. In other words, the theory can be applied to any material bodies undergoing homogeneous deformation. Nevertheless, it is appropriate to apply (14) to elastic or hyperelastic bodies for which $\mathbf{F}$ is a fundamental kinematic variable, and to apply

(15) to those material bodies with $\mathbf{L}$ as the fundamental kinematic variable. In this paper, we shall be concerned with a hyperelastic pseudo-rigid body which is defined by

$$\boldsymbol{\Sigma}^R = \mathrm{d}\sigma(\mathbf{F})/\mathrm{d}\mathbf{F}, \tag{21}$$

where σ is the stored-energy function. When a hyperelastic pseudo-rigid body is subjected to an internal constraint defined by $g(\mathbf{F}) = 0$, (21) should be replaced by

$$\boldsymbol{\Sigma}^R = \mathrm{d}\sigma(\mathbf{F})/\mathrm{d}\mathbf{F}+\mathbf{N}. \tag{22}$$

Here, as the reaction stress to the constraint, $\mathbf{N}$ has the form

$$\mathbf{N} = mq\,\mathrm{d}g/\mathrm{d}\mathbf{F}. \tag{23}$$

In (23) q is a scalar function to be determined by the balance equations of the body, and m denotes the mass of the body.

3. STATEMENT OF THE PROBLEM

Consider a flat hyperelastic membrane which has a hinged support point at b and has a movable point d on a fixed straight line co-planar with the membrane. The distance between the points b and d is arbitrarily adjustable. For simplicity of graphics, let the reference configuration of the membrane be circular. We use $\mathbf{X}_d$ and $\mathbf{x}_d$ to denote the reference and the present positions of the point d. Suppose that the point d is subjected to a dead load $\mathbf{G}$ parallel to the line, and that there is no frictional force at the points b and d. We take $\mathbf{T}$ and $\mathbf{H}$ to denote the external forces acting at the points b and d, respectively, which arise from contact of the membrane with the support point and the line, cf. Fig. 1. We will analyze the static deformation of the membrane in its own plane and preclude buckling out of the plane.

We let the origin o of the Euclidean space E be coincident with the fixed point b and choose an orthonormal basis $(o, \mathbf{i}_1, \mathbf{i}_2)$ such that the base vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ are normal and parallel to the fixed line, respectively. Let $\mathbf{X}_d$ be on the axis $\mathbf{i}_1$, as shown in Fig. 1. Then for the given problem, we can write

$$\mathbf{X}_d = L\mathbf{i}_1, \qquad \mathbf{x}_d = l_\alpha \mathbf{i}_\alpha, \tag{24}$$

$$\mathbf{G} = mG\mathbf{i}_2, \qquad \mathbf{T} = mT_\alpha \mathbf{i}_\alpha, \qquad \mathbf{H} = mH\mathbf{i}_1, \tag{25}$$

by applying the usual summation convention on repeated indices. Here, $\alpha = 1, 2$, L and l_1 are positive constants, while l_2 is a variable; and G, T_α and H determine the magnitude and direction of $\mathbf{G}$, $\mathbf{T}$ and $\mathbf{H}$, respectively.

To analytically seek the properties of the given structure, we regard the membrane as a pseudo-rigid body. Knowing that the deformation of the membrane under consideration is not homogeneous, we recognize that our results based on the theory of pseudo-rigid bodies will be approximate. With the given choice of the origin o and with the fixed point as the base point of the pseudo-rigid membrane, the static deformation and the balance equations can be represented as

$$\mathbf{x} = \mathbf{F}\mathbf{X}, \tag{26}$$

and

$$\mathbf{G}+\mathbf{T}+\mathbf{H} = \mathbf{0}, \qquad (\mathbf{G}+\mathbf{H})\otimes\mathbf{X}_d-\boldsymbol{\Sigma}^R = \mathbf{0}, \tag{27}$$

from (10) and (14), respectively, when the body force of the membrane is neglected and the inertia terms are removed.

If we write $\mathbf{F} = F_{\alpha\beta}\mathbf{i}_\alpha \otimes \mathbf{i}_\beta$, then it can be easily shown from (24) and (26) that

$$D := F_{11} = l_1/L, \qquad x := F_{21} = l_2/L, \tag{28}$$

where x measures the dimensionless displacement of the point d. We note that D is a constant and that $D > 0$ as follows from the definitions of l_1 and L.

In what follows, we will be concerned with the neo-Hookean hyperelastic membrane (Truesdell and Noll, 1965) characterized by

$$\sigma(\mathbf{F}) = m\mu(\mathbf{F}\cdot\mathbf{F}-2)/2, \qquad \det\mathbf{F} = 1, \tag{29}$$

where $\mu > 0$ is a material constant, and $(29)_2$ expresses the incompressibility of the material. We indicate that a three-dimensional body is incompressible if its volume is constant, while a two-dimensional body is incompressible if its area is conserved. It follows from (22), (23) and $g = \det\mathbf{F} - 1$ that

$$\boldsymbol{\Sigma}^R = m(\mu\mathbf{F} + q\mathbf{F}^{-T}), \tag{30}$$

where q is a scalar to be determined by the balance equations (27).

Finally, upon substitution of (24), (25), (28) and (30) into (27), we obtain the balance equations in the component form:

$$T_1 + H = 0, \qquad G + T_2 = 0, \qquad \mu D + qF_{22} - HL = 0, \tag{31}$$

$$\mu F_{12} - qx = 0, \qquad \mu x - qF_{12} - GL = 0, \qquad \mu F_{22} + qD = 0. \tag{32}$$

In order that the number of the equations is the same as that of the unknowns, we supply the condition

$$DF_{22} - xF_{12} = 1, \tag{33}$$

which follows from (28) and $(29)_2$. A general pseudo-rigid membrane has four degrees of freedom. However, since the external constraint $(28)_1$ and the internal constraint $(29)_2$ are introduced, the degrees of freedom of the deformation of the membrane are only two.

For the given problem, there is no need to decompose the deformation gradient $\mathbf{F}$ into the rotation tensor $\mathbf{R}$ and the stretch tensor $\mathbf{U}$ or $\mathbf{V}$. Of course, it is not difficult to obtain $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$ by the use of the formulae of Hoger and Carlson (1984), once $\mathbf{F}$ is found.

4. THE STATIC EQUILIBRIUM POSITIONS

In this section we will indicate the elastic behavior of the membrane described in Section 3 through the determination of the static equilibrium positions of the membrane.

It can be shown that (32) and (33) are equivalent to

$$p = x[1 - 1/(D^2 + x^2)^2], \tag{34}$$

$$F_{12} = -x/(D^2 + x^2), \qquad F_{22} = D/(D^2 + x^2), \qquad q = -\mu/(D^2 + x^2), \tag{35}$$

where the dimensionless moment p is defined by

$$p := GL/\mu. \tag{36}$$

Since once x is solved from (34), the other variables can be determined by (31) and (35), the variable x can be regarded as one defining the static deformation of the membrane. For this reason, we will call x satisfying (34) an equilibrium position of the membrane.

For further discussion, we write (34) in an alternative form:

$$x = (x-p)(D^2+x^2)^2. \tag{37}$$

It is trivial from (37) that $x = 0$ is an equilibrium position for all $D > 0$ if $p = 0$. To proceed, we consider the case in which p is small in the sense that $p \approx 0$, and seek the solution of (37) in the perturbation form:

$$x = x_0 + px_1 + O(p^2). \tag{38}$$

Upon substitution of (38) into (37), we get the perturbation equations:

$$x_0[1-1/(D^2+x_0^2)^2] = 0, \tag{39}$$

$$x_1 = (D^2+x_0^2)^2/[(D^2+x_0^2)^2+4(D^2+x_0^2)x_0^2-1]. \tag{40}$$

It is not difficult to see the following situation:

(a) If $D = 1$, (39) has only one solution $x_0 = 0$ but (40) is singular so that a perturbation solution of (37) does not exist.

(b) If $D > 1$, (39) has only one solution $x_0 = 0$ from which and (40), $x_1 = D^4/(D^4-1)$. Consequently, (37) has one perturbed solution,

$$x = pD^4/(D^4-1) + O(p^2). \tag{41}$$

(c) If $D < 1$, (39) has the three solutions:

$$x_0^A = 0, \qquad x_0^{B,C} = \pm\sqrt{1-D^2}, \tag{42}$$

from which and (40), $x_1^A = -D^4/(1-D^4)$, $x_1^{B,C} = 1/[4(1-D^2)]$. Consequently, (37) has three perturbed solutions:

$$x^A = -pD^4/(1-D^4) + O(p^2), \tag{43}$$

$$x^{B,C} = \pm\sqrt{1-D^2} + p/[4(1-D^2)] + O(p^2). \tag{44}$$

For the special case $p = 0$, it is apparent from (43) and (44) that the exact solution of (37) with $D < 1$ *is* (42), the last two of which together with (29) and (35) lead to

$$\sigma(\mathbf{F}) = 0, \qquad \Sigma^R(\mathbf{F}) = 0. \tag{45}$$

Here $(45)_2$ indicates that for $p = 0$, the equilibrium positions $x_0^{B,C}$ actually are those associated with the natural state of the membrane (Truesdell and Noll, 1965).

When p is arbitrary, a closed-form solution of the fifth-order algebraic equation (37) of x may not exist. But it is evident in view of (34) that p may be regarded as a function of x, although p itself is a parameter whereas x is a variable in the given problem. In fact, at a glance on (34), we find that for any $D > 0$, p is a single-valued, odd and smooth function of x, which has an asymptotic line $p = x$. From the oddness, suffice it to consider $x \geqslant 0$, in order to explore further properties of the structure.

It follows from differentiation of (34) with respect to x that

$$\begin{aligned} \mathrm{d}p/\mathrm{d}x &= 1-1/(D^2+x^2)^2+4x^2/(D^2+x^2)^3, \\ \mathrm{d}^2p/\mathrm{d}x^2 &= 12x(D^2-x^2)/(D^2+x^2)^4, \end{aligned} \tag{46}$$

which have the following properties:

(d) If $D > 0$, $\mathrm{d}^2p/\mathrm{d}x^2$ is zero for $x = 0$ or $x = D$; is positive for $0 < x < D$; is negative for $x > D$.

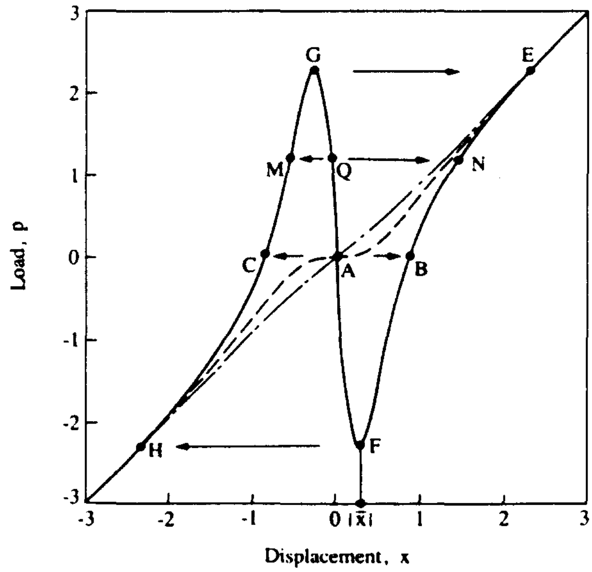

Fig. 2. Load p versus displacement x. ——, $D = 0.5$; — — —, $D = 1$; — - —, $D = 1.5$.

(e) If $D = 1$, $dp/dx = 0$ for $x = 0$, and $dp/dx > 0$ for all $x > 0$.
(f) If $D > 1$, $dp/dx > 0$ for all x.
(g) If $D < 1$,

$$dp/dx < 0 \quad \text{for } x = 0, \qquad dp/dx > 0 \quad \text{for } x \geqslant x_0^B, \tag{47}$$

where x_0^B is given by $(42)_2$. There exists only one real value $\bar{x}$ of x, satisfying

$$\bar{x}^2 + D^2 = (2D^2 + \sqrt{4D^4+1})^{1/3} + (2D^2 - \sqrt{4D^4+1})^{1/3}, \tag{48}$$

such that† $dp/dx|_{x=\bar{x}} = 0$. It is evident from (47) that $0 < \bar{x} < x_0^B$.

By gathering all the properties from (a) to (g) listed above and the anti-symmetry of the function $p(x)$, we are able to qualitatively draw the load–displacement curves in terms of p versus x for all D. Nevertheless, numerical examples may be helpful to explain the behavior of the structure here and later. For this reason, we plot (34) with $D = 0.5$, 1, 1.5 in Fig. 2. It is clear that in Fig. 2, the intersection of any straight line p = constant with these curves defines a static equilibrium position of the membrane. Thus the uniqueness or non-uniqueness of the static equilibrium positions of the membrane for any p can be observed from Fig. 2. Secondly, the figure shows that the curve of $D = 1$ is flat at $x = 0$. This is the reason why a perturbed solution in the form (38) does not exist for the structure with $D = 1$. Finally, it is found from Fig. 2 that if $D \gg 1$ the elastic structure comprised of the neo-Hookean pseudo-rigid membrane is almost linearly elastic. If $D = 1$, it is non-linearly and linearly elastic, when $|x| < 1$ and $|x| \geqslant 1$, respectively. If $D < 1$, it has negative and positive elastic tangent modulus when $|x| < |\bar{x}|$ and $|x| > |\bar{x}|$, respectively, where $|\bar{x}|$ satisfies (48). Therefore, one may use an elastic pseudo-rigid membrane to construct a "spring" with the behavior shown in Fig. 2.

While Fig. 2 gives the equilibrium positions of the membrane, the question as to the elastic stability of these equilibrium positions still remains. In the next section we will utilize the theory of elastic stability to study this question.

† After reduction of fractions to a common denominator, $dp/dx = 0$ is equivalent to $(D^2+x^2)^3 + 3(D^2+x^2) - 4D^2 = 0$, which as a Cardano's equation has only one real $\bar{x}$ root shown in (48).

5. ELASTIC STABILITY AND SNAPPING

As an application of the theory of elastic stability to a hyperelastic continuum, we will use *energy criterion I* (the adjacent method) stated as follows:

For a hyperelastic pseudo-rigid body, define an energy function:

$$V := \sigma - U, \tag{49}$$

where σ is the stored elastic energy of the body, and U is the potential of the external forces acting on the body. A static equilibrium position of the body is globally (locally) stable if and only if V is a minimum for all virtual deformations (for all virtual deformations near the equilibrium position).

Now for the given problem, it is not difficult to find that

$$\sigma = m\mu(F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2 - 2)/2,$$
$$U = \mathbf{G}\cdot x_d = mGl_2, \tag{50}$$

from (24), (25) and (29). For convenience, we define the dimensionless energy function

$$W := \frac{V}{m\mu}, \tag{51}$$

such that

$$W(x, F_{12}) = [D^2 + x^2 + F_{12}^2 + (1 + xF_{12})^2/D^2 - 2px - 2]/2, \tag{52}$$

with D and p as parameters, where (28), (33), (36), (49) and (50) have been used. Since for the membrane under consideration, the degrees of freedom of deformation are two, we will use δx and δF_{12} to denote its virtual deformation.

We first examine the case $p = 0$, for which $x = F_{12} = 0$ represents an equilibrium configuration of the structure with $D > 0$. For the structure with $D \geqslant 1$, it follows from (52) that

$$\begin{aligned} W(\delta x, \delta F_{12}) - W(0,0) &= [(D\delta x)^2 + (D\delta F_{12})^2 + (1 + \delta x\delta F_{12})^2 - 1]/(2D^2) \\ &\geqslant ([(\delta x)^2 + (\delta F_{12})^2 + (1 + \delta x\delta F_{12})^2 - 1]/(2D^2) \\ &= [(\delta x + \delta F_{12})^2 + (\delta x\delta F_{12})^2]/(2D^2) \geqslant 0, \end{aligned} \tag{53}$$

for any δx and δF_{12}. Hence, the equilibrium position $x = 0$ of the structure with $D \geqslant 1$ is globally stable in accordance with energy criterion I. Nevertheless, for $D < 1$, upon choosing $\delta x = aD = -\delta F_{12}$ with $0 < a^2 < 2(1 - D^2)/D^2$, we find that

$$\begin{aligned} W(\delta x, \delta F_{12}) - W(0,0) &= [2(aD)^2 + (1 - a^2D^2)^2/D^2 - 1/D^2]/2 \\ &= a^2[a^2D^2 - 2(1 - D^2)]/2 < 0. \end{aligned} \tag{54}$$

Thus, it follows from energy criterion I that $x = 0$ is an unstable equilibrium position of the structure with $D < 1$. It has been indicated that if $p = 0$, the structure with $D < 1$ has the other two equilibrium positions $x_0^{B,C}$ given by $(42)_2$. Let $F_{12}^{B,C}$ be the corresponding values of F_{12} at these positions. Then it is evident that

$$W(x_0^{B,C} + \delta x, F_{12}^{B,C} + \delta F_{12}) - W(x_0^{B,C}, F_{12}^{B,C}) \geqslant 0, \tag{55}$$

for any δx and δF_{12}, and

$$W(x_0^B, F_{12}^B) = W(x_0^C, F_{12}^C) = 0, \tag{56}$$

based on (45) and on the fact that the stored energy function W is non-negative. Thus, both of $x_0^{B,C}$ are globally and neutrally stable equilibrium positions of the membrane†.

Next, we examine the case $p \neq 0$. When p is small, it is mathematically difficult to determine the sign of variation of the energy function W (52) for all virtual deformations δx and δF_{12}. For finite p, we do not even have a closed-form expression for an equilibrium position of the membrane. For these reasons, we will use *energy criterion II* (the derivative method) state as follows:

A static equilibrium position of a hyperelastic pseudo-rigid body is locally stable if and only if the first- and the second-order variations of the energy function V (49) satisfy

$$\delta V = 0, \qquad \delta^2 V > 0, \tag{57}$$

for any virtual deformation near the equilibrium position. Furthermore, when $\delta V = \delta^2 V = 0$, i.e. when an equilibrium position is a critical point, then this position is locally stable if and only if

$$\delta^3 V = 0, \qquad \delta^4 V > 0. \tag{58}$$

For convenience, we will use W defined by (51) again to replace V in (57) and (58). It follows from (52) that $\delta W = 0$ and $\delta^2 W > 0$ hold if and only if

$$\partial W/\partial x = x + F_{12}(1 + xF_{12})/D^2 - p = 0,$$

$$\partial W/\partial F_{12} = F_{12} + x(1 + xF_{12})/D^2 = 0; \tag{59}$$

$$\partial^2 W/\partial x^2 = 1 + F_{12}^2/D^2 > 0, \qquad \partial^2 W/\partial F_{12}^2 = 1 + x^2/D^2 > 0, \tag{60}$$

$$\begin{aligned}\Delta &:= (\partial^2 W/\partial x^2)(\partial^2 W/\partial F_{12}^2) - (\partial^2 W/\partial x\, \partial F_{12})^2 \\ &= (1 + F_{12}^2/D^2)(1 + x^2/D^2) - (1 + 2xF_{12})^2/D^4 > 0.\end{aligned} \tag{61}$$

To proceed, we need the relation

$$\Delta = (1 + x^2/D^2)\,\mathrm{d}p/\mathrm{d}x, \tag{62}$$

which is derived from $(46)_1$, $(59)_2$ and (61). It is apparent that (59) and (60) hold for each equilibrium configuration, since (59) are equivalent to (34) and $(35)_1$. Thus, when energy criterion II is used, the stability of the equilibrium positions will be judged by (61) and by $\mathrm{d}p/\mathrm{d}x$, the sign of which can be easily determined from the properties (e)–(g) given in Section 4. In accordance with this argument, if $D = 1$, every $x \neq 0$ is locally stable; if $D > 1$, every x is locally stable; and if $D < 1$, every $|x| > |\bar{x}|$ is locally stable and every $|x| < |\bar{x}|$ is unstable. However, since $\Delta = 0$ at $x = 0$ for the case $D = 1$, and since $\Delta = 0$ at $x = \bar{x}$ for the case $D < 1$, where $\bar{x}$ satisfies (48), then $\delta^2 W$ may vanish for some virtual deformations in these two cases. Thus the higher-order variations of W should be examined for justification of stability in these two cases. It can be easily shown from (48) and (52) that

$$\delta^3 W = 0, \quad \delta^4 W = 12(\delta x)^2(\delta F_{12})^2/D^2 > 0, \quad \text{at } x = 0 \quad \text{for } D = 1, \tag{63}$$

$$\delta^3 W = 6x[\delta F_{12} - (D^2 + x^2)^{-1}\delta x]\delta x \delta F_{12}/D^2 \neq 0, \quad \text{at } x = \bar{x}, \quad \text{for } D < 1, \tag{64}$$

where δx and δF_{12} stand for a virtual deformation near the equilibrium positions. Equation

† An equilibrium position is neutrally stable if there exists a non-vanishing virtual deformation such that variation of the energy function W vanishes.

(63) implies stability of the position $x = 0$ for $D = 1$, while (64) means instability of the positions $\bar{x}$ for $D < 1$.

We summarize the conclusions obtained from the application of energy criterions I and II as follows:

(i) If $D \geqslant 1$, all the equilibrium positions are stable.

(ii) If $D < 1$, the equilibrium positions $|x| > |\bar{x}|$ are stable, whereas the positions $|x| \leqslant |\bar{x}|$ are unstable.

Now we are able to discuss snapping of the given structure. As usual, by snapping of a body, we mean the deformation of the body from an unstable equilibrium position to a stable equilibrium position when an infinitesimal disturbance is imparted to the body which is subjected to a constant load. It can be confirmed from (i) that for the structure with $D \geqslant 1$, since all the equilibrium positions are stable, no snapping of the membrane will take place. Subsequently, we will be concerned with the case $D < 1$ and explain snapping of the membrane by means of the analytic results and the curve with $D = 0.5$ in Fig. 2. Let the point d of the membrane be gradually displaced along the curve E–F with decrement of the load p. Before the point F is reached, all the static equilibrium positions are stable since the slope $\mathrm{d}p/\mathrm{d}x > 0$. However, when the point d is located at the equilibrium position F, it will snap through to H under any infinitesimal disturbance to the membrane which is subjected to a constant load p^F since $\bar{x}$ is an unstable equilibrium position. Since the curve is anti-symmetric, we can explain the same phenomenon when the point d is located at a position corresponding to a point in the curve H–G. Moreover, when the membrane is located at any position corresponding to a point in the curve G–F, say at the point Q, then it will snap through to either M or N under any infinitesimal disturbance to the membrane which is subjected to a constant load p^Q, since for the curve G–F, its slope $\mathrm{d}p/\mathrm{d}x < 0$. In contrast to the behavior of an elastic spherical cap shown by Brodland and Cohen (1987), no snap-back phenomenon will occur in the structure under consideration, because the load p is a single-valued function of the displacement x of the point d.

To show a possible application of the obtained results to an unstable elastic material point, we substitute (30) into $(20)_4$ so that the Cauchy force–moment of the neo-Hookean material is

$$\Sigma = m(\mu \mathbf{F}\mathbf{F}^T + q\mathbf{I}). \tag{65}$$

In particular, the shear force–moment $\Sigma_{12} = \Sigma_{21} = \Sigma \cdot \mathbf{e}_1 \otimes \mathbf{e}_2$ of the membrane is

$$\Sigma_{12} = m\mu Dx\left(1 - \frac{1}{(D^2+x^2)^2}\right), \tag{66}$$

by the use of (28), (35) and (65). It can be seen from (34) and (66) that the load p and the shear force–moment Σ_{12} only differ from each other by a constant coefficient. Thus, Σ_{12} has the same behavior as p. In the anti-shear problem of a neo-Hookean cylinder considered by Knowles (1989), the shear stress τ can be represented in terms of the shear deformation k by

$$\tau = M(k)k, \tag{67}$$

where M is called the secant modulus of shear. Knowles (1989) shows that the cylinder can sustain equilibrium shock waves if and only if its material in some region is elastically unstable. When there is no constitutive theory guiding the representation of the modulus $M(k)$, Knowles (1989) gives a $\tau \sim k$ curve in his Fig. 1(a), the profile of which is quite similar to that of $p \sim x$ with $D < 1$ in our Fig. 2, to represent an unstable elastic point. We notice that τ in (67) is the same as Σ_{12} in (66), and that both k in (67) and $x = F_{21}$ in (66) represent shear deformation. The form of (66) is a special instance of that of (67). As a result, we may use

$$M(k) = 1 - \frac{1}{(D^2+k^2)^2}, \qquad D < 1 \tag{68}$$

as a candidate for the representation of M in (67), since the relation (66) is obtained from a physical structure.

The present analysis is concerned with static deformation of the membrane. But snapping of the membrane usually involves dynamic deformation. In addition, a statically-stable equilibrium position may not be dynamically stable. Thus, further work based on dynamics is required to fully investigate the behavior of the membrane in the given structure.

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